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2000 J. Phys. A: Math. Gen. 33 6543

(http://iopscience.iop.org/0305-4470/33/37/306)

J. Phys. A: Math. Gen. **33** (2000) 6543–6547. Printed in the UK PII: S0305-4470(00)13753-9

Two-loop beta-functions of the sine–Gordon model

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Received 4 May 2000

Abstract. We recalculate the two-loop beta-functions in the two-dimensional sine–Gordon model in a two-parameter expansion around the asymptotically free point. Our results agree with those of Amit *et al* (Amit D J, Goldschmidt Y Y and Grinstein G 1980 *J. Phys. A: Math. Gen.* **13** 585).

In this paper we recalculate the two-loop beta-function coefficients in the two-dimensional sine–Gordon (SG) model in a two-parameter perturbative expansion around the asymptotically free (AF) point. The study of the SG model in the vicinity of this point is especially important since this region is used in the description of the Kosterlitz–Thouless (KT) phase transition in the two-dimensional O(2) nonlinear σ-model, better known as the XY model†. This was the motivation of the authors of [2], who have undertaken a systematic study of perturbation theory in a two-parameter expansion around the AF point. They calculated the renormalization group (RG) beta-functions up to the two-loop coefficients. The beta-function coefficients were also calculated in [3] by a completely different technique based on string theory. The results found in [3] differ from those of [2] at the two-loop level. The question of two-loop beta-function coefficients was considered also in [4] for a class of generalized SG models. The results, when specialized to the case of the ordinary SG model, agree with those of [3], but disagree with those of [2]. In [5] the short-distance expansion of some SG correlation functions were calculated using conformal perturbation theory. This allowed the extraction of the one- and two-loop beta-function coefficients around the AF point. The resulting two-loop beta-functions differ from all the previous results.

In view of the role the SG model is playing in the description of the KT phase transition it is very important to know the correct two-loop beta-function coefficients. The purpose of this paper is to show that, in fact, the two-loop results of Amit *et al* [2] are the correct ones. We show this first by comparing the SG beta-function to known results in the chiral Gross–Neveu model [6], which is known to be equivalent to the SG model at its AF point. We also check the beta-function coefficients by considering the renormalization of $2n$-point functions of exponentials of the SG field.

Following [2] we consider the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m_0^2}{2}\phi^2 + \frac{\alpha_0}{\beta_0^2 a^2}[1 - \cos(\beta_0\phi)] \tag{1}$$

† For a review of the SG description of the KT theory, see [1].

where m_0 is an IR regulator mass and a is the UV cutoff (of dimension length). UV regularized correlation functions are calculated by using

$$G_0(x) = \frac{1}{2\pi} K_0 \left(m_0 \sqrt{x^2 + a^2} \right) \tag{2}$$

where K_0 is the modified Bessel function, as the ϕ propagator. Our strategy is slightly different from [2], who really considered the renormalization of the massive SG model (1) of mass m_0. We treat m_0 as an IR regulator mass and consider IR-stable physical quantities for which we can take the limit $m_0 \to 0$ already at the UV regularized level (before UV renormalization).

The model (1) is renormalizable in a two-parameter perturbative expansion around the point corresponding to the couplings $\alpha_0 = 0$, $\beta_0^2 = 8\pi$. Writing

$$\beta_0^2 = 8\pi(1 + \delta_0) \tag{3}$$

the two bare expansion parameters are α_0 and δ_0 and physical quantities can be made UV finite by the renormalizations

$$\alpha_0 = Z_\alpha \alpha \qquad Z_\alpha = 1 + g_1 \delta \ell + \alpha^2(\overline{g}_2 \ell^2 + g_2 \ell) + \delta^2(\overline{g}_3 \ell^2 + g_3 \ell) + \cdots \tag{4}$$

$$1 + \delta_0 = Z_\phi^{-1}(1 + \delta) \qquad Z_\phi = 1 + f_1 \alpha^2 \ell + \alpha^2 \delta(\overline{f}_2 \ell^2 + f_2 \ell) + \cdots \tag{5}$$

where α and δ are the renormalized couplings and $\ell = \ln \mu a$ with μ an arbitrary renormalization point. The dots stand for terms of higher order in perturbation theory and the numerical coefficients g_1, f_1 etc can be calculated by renormalizing correlation functions. The results of Amit *et al* [2] are

$$f_1 = \tfrac{1}{32} \qquad g_1 = -2 \qquad f_2 = -\tfrac{3}{32} \qquad g_2 = -\tfrac{5}{64} \qquad g_3 = 0 \tag{6}$$

those of [3] and [4] are

$$f_1 = \tfrac{1}{32} \qquad g_1 = -2 \qquad f_2 = -\tfrac{1}{32} \qquad g_2 = -\tfrac{1}{32} \qquad g_3 = 0 \tag{7}$$

and finally [5] found

$$f_1 = \tfrac{1}{32} \qquad g_1 = -2 \qquad f_2 = -\tfrac{1}{32} \qquad g_2 = -\tfrac{1}{16} \qquad g_3 = 0. \tag{8}$$

We see that the one-loop coefficients are the same but not all two-loop coefficients agree. The subject of this paper is to recalculate these numbers.

The RG beta-functions can be calculated by solving the equations

$$\mathcal{D}\alpha = \mathcal{D}\delta = 0 \tag{9}$$

where, as usual, the RG operator is defined by

$$\mathcal{D} = -a \frac{\partial}{\partial a} + \beta_\alpha(\alpha_0, \delta_0) \frac{\partial}{\partial \alpha_0} + \beta_\delta(\alpha_0, \delta_0) \frac{\partial}{\partial \delta_0}. \tag{10}$$

One finds

$$\beta_\alpha = -g_1 \alpha_0 \delta_0 - g_2 \alpha_0^3 - g_3 \alpha_0 \delta_0^2 + \cdots \tag{11}$$

$$\beta_\delta = f_1 \alpha_0^2 + (f_1 + f_2) \alpha_0^2 \delta_0 + \cdots. \tag{12}$$

It is well known that, in the case of several couplings, the higher beta-function coefficients are not all scheme independent. Indeed, considering the perturbative redefinitions

$$\tilde{\alpha}_0 = \alpha_0 + c_1 \alpha_0 \delta_0 + \cdots \qquad \tilde{\delta}_0 = \delta_0 + c_2 \alpha_0^2 + \cdots \tag{13}$$

one finds that in addition to the one-loop coefficients f_1 and g_1 only the following two two-loop coefficient combinations are invariant:

$$g_3 \qquad J = 2g_2 - f_2. \tag{14}$$

The RG analysis with two couplings can be made similar to the case of a single coupling by changing the variables from α_0 and δ_0 to the pair Q and δ_0, where Q is an RG invariant (solution of the $\mathcal{D}Q = 0$ equation), given in perturbation theory by

$$Q = f_1\alpha_0^2 + g_1\delta_0^2 + 2g_2\alpha_0^2\delta_0 + F_2\delta_0^3 + \cdots \tag{15}$$

where $F_2 = \frac{2}{3}g_3 - \frac{2}{3}g_1 + \frac{2g_1}{3f_1}J$. Now Q, being an RG invariant, can almost be treated as if it were a numerical constant and δ_0 as the 'true' coupling. The beta-function in these variables is

$$\beta(\delta_0, Q) = Q + 2\delta_0^2 + AQ\delta_0 + B\delta_0^3 + \cdots \tag{16}$$

where $A = 1 - J/f_1$ and $B = 2(A - g_3)/3$.

It is well known that the SG model can also be formulated in terms of two fermion fields, interacting with a chirally symmetric current–current interaction [1]. A special case of the two-fermion model corresponds to the $SU(2)$-symmetric chiral Gross–Neveu model. This correspondence is evident in the bootstrap aproach, since the SG S-matrix in the limit $\beta_0 \to \sqrt{8\pi}$ becomes the $SU(2)$ chiral Gross–Neveu S-matrix. This AF model has to correspond to one of the possible RG trajectories in the two-parameter SG language. It is easy to see that it has to be the $Q = 0$ trajectory, since this is the only trajectory going through the origin ($\delta_0 = \alpha_0 = 0$) of the parameter space. More precisely, the chiral Gross–Neveu model must correspond to the negative half of the $Q = 0$ trajectory, which is a UV AF trajectory. Making the identification

$$\delta_0 = -\frac{1}{\pi}g^2 \tag{17}$$

where g is the coupling of the $SU(2)$ Gross–Neveu model, the Gross–Neveu beta-function becomes

$$\beta(g) = -\frac{1}{\pi}g^3 + \frac{B}{2\pi^2}g^5 + \cdots. \tag{18}$$

Using the results of Amit *et al* (equation (6)), $B = 2$, and using the results of [3] and [4] (equation (7)), $B = 4/3$, and finally $B = 8/3$ if we trust [5] (equation (8)). Comparing (18) to the results of the beta-function calculations performed directly in the fermion language [6] we see that the correct Gross–Neveu beta-function is reproduced if $B = 2$. Thus the two-loop results of Amit *et al* [2] are correct after all! This was the observation† that served as our motivation for the present study. The correctness of the two-loop Gross–Neveu beta-function coefficient has been checked by studying the system in the presence of an external field [7]. Using this method the value of this coefficient can be read off from the bootstrap S-matrix and the results are in agreement with [6].

We now turn to the explicit calculation of the renormalization parameters (4), (5). The first quantity we consider is the two-point function of the $U(1)$-current $J_\mu = \mathrm{i}\frac{\beta_0}{2\pi}\epsilon_{\mu\nu}\partial_\nu\phi$,

$$\langle J_\mu(x)J_\nu(y)\rangle = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\left(\frac{p_\mu p_\nu}{p^2} - \delta_{\mu\nu}\right)\mathrm{e}^{\mathrm{i}p(x-y)}I(p). \tag{19}$$

The advantage of considering this physical quantity is that it is IR stable. Putting $m_0 = 0$ we find

$$I(p) = \frac{2}{\pi}\left\{1 + \delta_0 + \frac{\alpha_0^2}{32}\left(\ln pa + K + \frac{1}{2}\right) + \frac{\alpha_0^2\delta_0}{16}(\ln pa + K)^2 + \cdots\right\} \tag{20}$$

† We thank P Forgács who made this observation first and called our attention to it.

where $K = -\Gamma'(1) - 1 - \ln 2$. Since the current is conserved there is no operator renormalization required here and (20) must become finite after the substitutions (4), (5). From this requirement we obtain

$$f_1 = \tfrac{1}{32} \qquad g_1 = -2 \qquad f_2 = -\tfrac{3}{32}. \tag{21}$$

To determine the remaining two-loop coefficients g_2 and g_3 we have to calculate Z_α, the renormalization constant corresponding to α_0. For this purpose we need a quantity with a perturbative series starting at $\mathcal{O}(\alpha_0)$. We have chosen the $2n$-point correlation function

$$X = \langle \mathcal{A}(x_1) \ldots \mathcal{A}(x_{2n}) \rangle \tag{22}$$

where

$$\mathcal{A}(x) = \left(\frac{1}{a}\right)^{\frac{1}{2n^2}} \exp\left(\frac{\mathrm{i}\beta_0}{2n}\phi(x)\right). \tag{23}$$

Although, in contrast to the Noether current, the operator (23) needs to be renormalized, for large enough n the dimension of (23) is so small that there is no operator mixing and the operator renormalization constant can simply be determined from the correlation function

$$Y = \langle \mathcal{A}(x_1) \ldots \mathcal{A}(x_n) \mathcal{A}^*(y_1) \ldots \mathcal{A}^*(y_n) \rangle. \tag{24}$$

A second-order calculation gives

$$\begin{aligned} Y = M^{(\frac{1}{n^2})} \Bigg\{ 1 &+ \frac{\delta_0}{n^2} L + \frac{\delta_0^2}{2n^4} L^2 + \frac{\alpha_0^2}{64n^3} L^2 \\ &+ L \left(\frac{\alpha_0^2}{64n^2} - \frac{\alpha_0^2}{128} \left[W\left(\frac{1}{n}\right) + W\left(-\frac{1}{n}\right) \right] \right) + \cdots \Bigg\} \end{aligned} \tag{25}$$

where

$$M = \frac{\prod_{i<j} |x_i - x_j| \prod_{k<l} |y_k - y_l|}{\prod_{i,k} |x_i - y_k|} \tag{26}$$

$L = \ln Ma^n$ and the dots stand for finite $\mathcal{O}(\alpha_0^2)$ terms as well as higher-order terms. $W(\mu)$ is defined by

$$W(\mu) = -1 + \int_0^1 \mathrm{d}z\, z^\mu F(\mu, \mu; 1; z) + \int_0^1 \frac{\mathrm{d}z}{z^2} [F(\mu, \mu; 1; z) - 1 - \mu^2 z] \tag{27}$$

where $F(\alpha, \beta; \gamma; z)$ is the standard hypergeometric function. Equation (25) can be made finite by the renormalization $Y_R = Z_{2n} Y$, where

$$Z_{2n} = 1 - \frac{1}{n}\ell\delta + \frac{1}{2n^2}\ell^2\delta^2 + \frac{1}{64n}\ell^2\alpha^2 + k_1\ell\alpha^2 + \cdots \tag{28}$$

with

$$k_1 = -\frac{1}{64n} + \frac{n}{128}\left[W\left(\frac{1}{n}\right) + W\left(-\frac{1}{n}\right) \right]. \tag{29}$$

For the $2n$-point function X a second-order calculation gives

$$X = \frac{\alpha_0}{16\pi} N^{(\frac{1}{n^2})} F \left\{ 1 + \delta_0 \Psi + \frac{1}{2}\delta_0^2 \Psi^2 + \frac{n\alpha_0^2}{128n + 64} \Psi \left[\Psi + 4 + \frac{1}{n} - nW\left(\frac{1}{n}\right) \right] + \cdots \right\} \tag{30}$$

where

$$N = \prod_{i<j} |x_i - x_j| \qquad F = \int \mathrm{d}^2 z \frac{1}{\prod_i |z - x_i|^{\frac{2}{n}}} \tag{31}$$

and

$$\Psi = -1 + \frac{1}{n^2}\ln(Na^{-n(2n-1)}) - \frac{2}{nF}\sum_j \int d^2z \frac{\ln|\frac{z-x_j}{a}|}{\prod_i |z-x_i|^{\frac{2}{n}}}. \tag{32}$$

In (30) the dots represent finite terms of $\mathcal{O}(\alpha_0^2)$ and $\mathcal{O}(\delta_0^2)$ as well as higher terms. Renormalizing X by requiring $X_R = Z_{2n}X$ to be finite after coupling constant renormalization gives

$$g_3 = 0 \qquad \text{and} \qquad g_2 = -\frac{1}{16} + \frac{n}{128}\left[W\left(\frac{1}{n}\right) - W\left(-\frac{1}{n}\right)\right]. \tag{33}$$

At first sight g_2 seems to be n dependent, which would mean that the $2n$-point function (22) cannot really be made finite with wavefunction plus coupling constant renormalization. Luckily, however, one can see that using the identity

$$W(\mu) - W(-\mu) = -2\mu \qquad (|\mu| < 1) \tag{34}$$

satisfied by the hypergeometric function, g_2 is equal to the n-independent constant $-\frac{5}{64}$. Moreover, (33) together with (21) reproduce (6), the results of [2]. The nontrivial cancellation of the n dependence makes us more confident that these are the correct two-loop coefficients.

Acknowledgments

We would like to thank P Forgács for calling our attention to the fact that the results of [2] reproduce the correct two-loop coefficients for the chiral Gross–Neveu model. We thank R Konik for correspondence.

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